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A NEW RENORMALIZATION METHOD FOR THE ASYMPTOTIC SOLUTION OF WEAKLY NONLINEAR VECTOR SYSTEMS*

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Abstract. This paper considers the asymptotic integration of a special class of initial value problems involving a nonlinear regular perturbation scaled by a small parameter $\epsilon > 0$. For $t = \mathcal{O}(1/\epsilon)$, these problems were classically solved using either the method of averaging or of multiple scales to remove secular terms that arise in the natural power series procedure. Our new ansatz is straightforward and effective. Moreover, it indicates when problems might occur in providing the asymptotic solution on very long time intervals. Other closely related problems are also attacked using renormalization.

Key words. oscillations, asymptotics, renormalization

AMS subject classifications. 34C15, 34E10

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Background: The rise of secular terms. We shall seek the asymptotic solution $x_\epsilon(t)$ of the initial value problem for the weakly nonlinear nearly autonomous vector system

$$\dot{x} = Mx + \epsilon N(x, t, \epsilon) \tag{1}$$

on the semi-infinite time interval $t \geq 0$ as the small positive parameter ϵ tends to zero. Such problems and their generalizations describe numerous electrical, mechanical, and biological oscillations. Indeed, the asymptotic solution of related boundary value problems for partial differential equations remains of substantial interest and importance. Without further hypotheses, however, one can't predict the time interval on which the solution remains bounded. We shall assume that the matrix M has only imaginary eigenvalues, that the fundamental matrix e^{Mt} for the unperturbed problem has a period $p > 0$, and that the vector N is smooth in its three arguments and p-periodic in t. We could even assume that M is a diagonal matrix having a spectral decomposition $M = iV\Lambda V^{-1}$ with a real diagonal matrix Λ and introduce the transformation $\widetilde{x} = Vx$.

By variation of parameters,

$$z_\epsilon(t) = e^{-Mt} x_\epsilon(t) \tag{2}$$

will satisfy the transformed system

$$\dot{z} = \epsilon f(z, t, \epsilon), \tag{3}$$

analogous to (1) with $M = 0$, for the p-periodic forcing

$$f(z, t, \epsilon) \equiv e^{-Mt} N(e^{Mt} z, t, \epsilon). \tag{4}$$

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Indeed, we will say the system (3) is in standard form. Moreover, anticipating that (3) will have a nearly constant solution for bounded times, setting

$$x_\epsilon(t) = e^{Mt}(x(0) + \epsilon u(t, x(0), \epsilon)) \qquad \text{or} \qquad z_\epsilon(t) = x(0) + \epsilon u(t, x(0), \epsilon) \tag{5}$$

shows that the scaled correction vector u will satisfy the nearly linear initial value problem

$$\dot{u} = f(x(0) + \epsilon u, t, \epsilon) \tag{6}$$

on some interval $t \geq 0$ with $u(0, x(0), \epsilon) = 0$.

The natural starting point for obtaining an asymptotic solution $x_\epsilon(t)$ of (1) or $z_\epsilon(t)$ to (3) is to introduce the regular power series expansion

$$u(t, x(0), \epsilon) = u_0(t, x(0)) + \epsilon u_1(t, x(0)) + \epsilon^2 u_2(t, x(0)) + \cdots \tag{7}$$

for u, determining its terms u_j uniquely and successively by equating coefficients of like powers of ϵ in the differential equation (6) and the initial condition. Thus, the u_j's must satisfy the resulting sequence of linear initial value problems

$$\begin{cases} \dot{u}_0 = f(x(0), t, 0), \qquad u_0(0) = 0, \\ \dot{u}_1 = f_x(x(0), t, 0)u_0 + f_\epsilon(x(0), t, 0), \qquad u_1(0) = 0, \\ \dot{u}_2 = \ f_x(x(0), t, 0)u_1 + \frac{1}{2}[f_{xx}(x(0), t, 0)u_0 + 2f_{x\epsilon}(x(0), t, 0)]u_0 \\ \qquad + \frac{1}{2}f_{\epsilon\epsilon}(x(0), t, 0), \qquad u_2(0) = 0, \\ \text{etc.,} \end{cases} \tag{8}$$

and thus integrating successively immediately provides the coefficients

$$u_0(t, x(0)) = \int_0^t f(x(0), s, 0)ds, \tag{9}$$

$$u_1(t, x(0)) = \int_0^t [f_x(x(0), s, 0)u_0(s, x(0)) + f_\epsilon(x(0), s, 0)]ds, \tag{10}$$

etc., in (7). Using standard Gronwall inequality arguments (cf. Smith (1985) or Murdock (1991)), it becomes clear that the regular power series (7) provides the asymptotic solution $x_\epsilon(t)$ as $\epsilon \to 0$ on bounded t intervals.

Recall, however, that Lagrange, Laplace, Poincaré, and other developers of celestial mechanics knew that ordinary *resonance* implies that these u_j's generally contain secular terms that grow as polynomials in t of degree $j + 1$. This implies that the expansion (7) then loses its asymptotic validity on long time intervals since the terms $\epsilon^{j+1}u_j(t)$ of ϵu all attain the same asymptotic order when $t = \mathcal{O}(1/\epsilon)$. For this reason, the power series (7) was called *naive* by Chen, Goldenfeld, and Oono (1996). Many asymptotic methods have been developed to deal with this dilemma. The most important classical techniques are the Krylov–Bogoliubov *averaging* method, largely developed in Kiev in the 1930s (cf. Bogoliubov and Mitropolsky (1961)), and two-timing or the method of *multiple scales*, developed at Caltech in the 1960s (cf. Kevorkian and Cole (1996), but note independent early contributions of Kuzmak (1959), Cochran (1962), and Mahony (1962)). Our work relates closely to the *renormalization group* method of Chen, Goldenfeld, and Oono (1996) and the *invariance* *condition* method of Woodruff (1993, 1995), though averaging and multiple scale concepts remain essential to its development. Readers should note that Oono (2000) and Nozaki and Oono (2001) simplify the earlier renormalization group method and that Jarrad (2001) includes a promising *variational perturbation theory.* Chen, Goldenfeld, and Oono (1996) began their paper by suggesting that "the practice of asymptotic analysis is something of an art." Like them, we seek to show that "the renormalization group approach sometimes seems to be more efficient and accurate than standard methods in extracting global information from the perturbation expansion."

Simple resonance considerations show that u_0 will grow like a multiple of t as $t \to \infty$ if and only if its known forcing $f(x(0), t, 0)$ has a nonzero *average*

$$\langle f(x(0), t, 0)\rangle \equiv \frac{1}{p}\int_0^p f(x(0), s, 0)ds, \tag{11}$$

which conveniently coincides with the leading term in its Fourier series expansion on $0 \le t \le p$. Indeed, if we split $f(x(0), t, 0)$ into its average and supplementary *fluctuating zero-average part*

$$\{f(x(0), t, 0)\} = f(x(0), t, 0) - \langle f(x(0), t, 0)\rangle, \tag{12}$$

the response u_0 analogously splits into the sum

$$u_0(t, x(0)) = ta_0(x(0)) + U_0(x(0), t) \tag{13}$$

of its corresponding secular part $a_0 t$, with the average

$$a_0(x(0)) \equiv \langle f(x(0), t, 0)\rangle \tag{14}$$

as a coefficient and with the *bounded* secular-free part

$$U_0(x(0), t) \equiv \int_0^t \{f(x(0), s, 0)\}ds. \tag{15}$$

Substituting (13) into (10), integrating by parts, and splitting f_x into its average and fluctuating parts, we next get

$$\begin{aligned} u_1(t, x(0)) &= \left(\int_0^t sf_x(x(0), s, 0)ds\right) a_0(x(0)) \\ &\qquad + \int_0^t [f_x(x(0), s, 0)U_0(x(0), s) + f_\epsilon(x(0), s, 0)]ds \\ &= \left(\frac{1}{2}t^2\langle f_x(x(0), t, 0)\rangle + t\frac{\partial U_0}{\partial x}(x(0), t)\right) a_0(x(0)) \\ &\qquad + \int_0^t \Big[f_x(x(0), s, 0)U_0(x(0), s) + f_\epsilon(x(0), s, 0) \\ &\qquad\qquad - \frac{\partial U_0}{\partial x}(x(0), s)a_0(x(0))\Big]\, ds. \end{aligned}$$

Thus, u_1 has the predicted polynomial form

$$\begin{aligned} u_1(t, x(0)) &= \frac{1}{2}t^2\langle f_x(x(0), t, 0)\rangle a_0(x(0)) \\ &\quad + t\left[a_1(x(0)) + \frac{\partial U_0}{\partial x}(x(0), t)a_0(x(0))\right] + U_1(x(0), t), \end{aligned} \tag{16}$$

where the coefficients involve the average

$$a_1(x(0)) \equiv \left\langle f_x(x(0),t,0)U_0(x(0),t) + f_\epsilon(x(0),t,0) - \frac{\partial U_0}{\partial x}(x(0),t)a_0(x(0))\right\rangle$$

and the supplementary term

$$U_1(x(0),t) \equiv \int_0^t \bigg\{ f_x(x(0),s,0)U_0(x(0),s) + f_\epsilon(x(0),s,0) - \frac{\partial U_0}{\partial x}(x(0),s)a_0(x(0))\bigg\}ds \tag{17}$$

that remains bounded for all $t \geq 0$. (Corresponding higher-order first and final coefficients a_j and U_j won't be so directly linked as when $j = 0$ and 1.)

Continuing in this manner, however, we ultimately learn that a bounded asymptotic solution $z_\epsilon(t)$ results on a longer time interval from using only the bounded secular-free (or so-called *bare*) part

$$x(0) + \epsilon U_0(x(0),t) + \epsilon^2 U_1(x(0),t) + \epsilon^3(\cdots)$$

of the regular power series for z_ϵ. Further, we must generally replace the initial vector $x(0)$ by a time-varying amplitude $A_\epsilon(\tau)$ depending on the slow-time

$$\tau = \epsilon t \tag{18}$$

and found by integrating the initial value problem

$$\frac{dA_\epsilon}{d\tau} = a_0(A_\epsilon) + \epsilon a_1(A_\epsilon) + \mathcal{O}(\epsilon^2), \quad A_\epsilon(0) = x(0),$$

on (possibly unbounded) τ intervals where its solution remains bounded. (Observe that one might interpret the replacement of $x(0)$ by the slowly varying $A_\epsilon(\tau)$ as finding an envelope of solutions (cf. Ei, Fujii, and Kunihiro (2000)). Likewise, one could be motivated by Whitham's success in using slowly varying functions to asymptotically solve nonlinear partial differential equations (cf. Whitham (1974) and Debnath (1997)) or by the use of related *amplitude equations* in stability theory (cf. Coullet and Spiegel (1983), Eckhaus (1992), and Promislow (2001)). The basic ploy is to eliminate the secular terms from the naive expansion (7). Moreover, observe that replacing $x(0)$ by $A_\epsilon(\tau)$ also makes our leading-order approximation $e^{Mt}A_\epsilon(\tau)$ to $x_\epsilon(t)$ richer, although such an improvement will not be asymptotically noticeable when t is only finite. We admit that this simple renormalization result still remains largely unmotivated, but we shall now obtain it by using an effective ansatz that could be applied more generally (e.g., in asymptotically stable contexts where M is a stable matrix and $\frac{1}{p}\int_0^p f(x(0),s,0)ds$ converges as $p \to \infty$, allowing us to take an infinite p to again define the averaged equation satisfied by the limiting $A_0(\tau)$. When $M = \left(\begin{smallmatrix} -Q & 0 \\ 0 & iR \end{smallmatrix}\right)$ for an exponentially decaying matrix e^{-Qt} and a periodic e^{iRt}, we would use such a long-time average to approximate the first components of x).

The basic ansatz. We shall begin anew to solve (1) by directly introducing the multiple-scale *ansatz*

$$\begin{cases} x_\epsilon(t) = e^{Mt}z_\epsilon(t) \equiv e^{Mt}[A_\epsilon(\tau) + \epsilon U(A_\epsilon(\tau),t,\epsilon)] \\ \text{or} \\ z_\epsilon(t) = A_\epsilon(\tau) + \epsilon U(A_\epsilon(\tau),t,\epsilon) \end{cases} \tag{19}$$

corresponding to the *bare* expansion of Chen, Goldenfeld, and Oono (1996). It can be motivated for problem (1), at least for τ finite, by substituting (19) into the differential equation (3). Using the chain rule, we get

$$\frac{1}{\epsilon}\frac{dz_\epsilon}{dt} = \left(I + \epsilon\frac{\partial U}{\partial A_\epsilon}\right)\frac{dA_\epsilon}{d\tau} + \frac{\partial U}{\partial t} = f. \tag{20}$$

We now split $\frac{\partial U}{\partial A_\epsilon}$ and f into sums of their average and mean-free fluctuating parts, respectively, using the leading term and the supplementary sum of their Fourier expansions on $0 \le t \le p$, and asking that A_ϵ account for the nonzero average terms

$$\left(I + \epsilon\left\langle\frac{\partial U}{\partial A_\epsilon}\right\rangle\right)\frac{dA_\epsilon}{d\tau} = \langle f\rangle$$

in (20), while the correction ϵU to A_ϵ in (19) handles its remaining terms

$$\epsilon\left\{\frac{\partial U}{\partial A_\epsilon}\right\}\frac{dA_\epsilon}{d\tau} + \frac{\partial U}{\partial t} = \{f\}$$

with zero averages. (Readers might indeed recall that an analogous decomposition occurs in the early comparison by Morrison (1966) of averaging and two-timing.) Thus, A_ϵ should satisfy the autonomous system

$$\begin{aligned}\frac{dA_\epsilon}{d\tau} &= \left(I + \epsilon\left\langle\frac{\partial U}{\partial A_\epsilon}(A_\epsilon(\tau), t, \epsilon)\right\rangle\right)^{-1}\langle f(A_\epsilon + \epsilon U(A_\epsilon, t, \epsilon), t, \epsilon)\rangle \\ &\equiv a\ (A_\epsilon, \epsilon) \\ &= \langle f(A_\epsilon, t, 0)\rangle \\ &\quad + \epsilon\Big[\langle f_x(A_\epsilon, t, 0)U_0(A_\epsilon, t) + f_\epsilon(A_\epsilon, t, 0)\rangle \\ &\qquad - \left\langle\frac{\partial U_0}{\partial A_\epsilon}(A_\epsilon, t)\right\rangle\langle f(A_\epsilon, t, 0)\rangle\Big] + \epsilon^2(\cdots)\end{aligned} \tag{21}$$

and the initial condition $A_\epsilon(0) = x(0)$, while U must satisfy

$$\frac{\partial U}{\partial t} = \{f\} - \epsilon\left\{\frac{\partial U}{\partial A_\epsilon}\right\}a(A_\epsilon, \epsilon)$$

and the trivial initial condition $U(A_\epsilon(\tau), 0, \epsilon) = 0$. We shall call the differential equation (21) the *amplitude* or *first level RG (renormalization group) equation*, noting that an analogous evolution equation results when one uses the higher-order method of averaging. The asymptotic integration of (21) on $\tau \ge 0$ is the appropriate candidate problem to replace the integration of (1) after t becomes unbounded. In these differential equations for A_ϵ and U, the times t and τ are taken to be independent variables, as is typical in two-timing. Integrating the latter equation with respect to t shows that U must satisfy the integral equation

$$\begin{aligned}U(A_\epsilon(\tau), t, \epsilon) &= \int_0^t \{f(A_\epsilon(\tau) + \epsilon U(A_\epsilon(\tau), s, \epsilon), s, \epsilon)\}ds \\ &\quad - \epsilon\int_0^t \left\{\frac{\partial U}{\partial A_\epsilon}(A_\epsilon(\tau), s, \epsilon)\right\}ds\, a(A_\epsilon(\tau), \epsilon).\end{aligned} \tag{22}$$

That we have obtained the compact formulae (21) and (22) to all orders in ϵ is quite helpful, though we naturally next employ power series methods to get more explicit asymptotic results for bounded τ values. Note, in particular, that $\frac{\partial U}{\partial t}$ has a zero average, so its integral U in (22) will be bounded whenever the amplitude $A_\epsilon(\tau)$ is. The resulting power series expansion

$$U(A_\epsilon(\tau), t, \epsilon) = U_0(A_\epsilon(\tau), t) + \epsilon U_1(A_\epsilon(\tau), t) + \epsilon^2(\cdots) \tag{23}$$

has coefficients successively and unambiguously given by

$$\begin{aligned} U_0(A_\epsilon(\tau), t) &= \int_0^t \{f(A_\epsilon(\tau), s, 0)\} ds, \\ U_1(A_\epsilon(\tau), t) &= \int_0^t \{f_x(A_\epsilon(\tau), s, 0) U_0(A_\epsilon(\tau), s) + f_\epsilon(A_\epsilon(\tau), s, 0)\} ds \\ &\quad - \int_0^t \left\{ \frac{\partial U_0}{\partial A_\epsilon}(A_\epsilon(\tau), s) \right\} ds \, a_0(A_\epsilon(\tau)), \end{aligned}$$

etc., corresponding to the functions previously obtained in (15) and (17) for the non-secular parts of the naive expansion (7). Note that U is p-periodic in t.

The remaining, and still formidable, task is to obtain the asymptotic solution of the initial value problem (21) for the slowly varying amplitude A_ϵ on time intervals where it will determine a bounded solution x_ϵ or z_ϵ via (19) and (22). We naturally first seek $A_\epsilon(\tau)$ as a power series

$$A_\epsilon(\tau) = A_0(\tau) + \epsilon A_1(\tau) + \epsilon^2 A_2(\tau) + \cdots, \tag{24}$$

where (21) implies that its limit A_0 must satisfy the limiting nonlinear problem

$$\frac{dA_0}{d\tau} = a_0(A_0) \equiv \frac{1}{p} \int_0^p f(A_0, s, 0) ds, \quad A_0(0) = x(0), \tag{25}$$

exactly as in classical averaging, while the next term A_1, for example, must satisfy a linearized problem

$$\frac{dA_1}{d\tau} = \frac{da_0(A_0)}{dA_0} A_1 + a_1(A_0), \quad A_1(0) = 0.$$

The uniqueness of A_0 implies the invertibility of the Jacobian matrix $\frac{\partial A_0}{\partial x(0)}$, which satisfies the homogeneous linear matrix system as long as A_0 remains defined. If A_0 ever blows up, we must naturally limit our interval of approximation. Using a variation of parameters, then,

$$A_1(\tau) = \frac{\partial A_0(\tau)}{\partial x(0)} \int_0^\tau \left(\frac{\partial A_0(s)}{\partial x(0)} \right)^{-1} a_1(A_0(s)) ds \tag{26}$$

and later terms A_j also follow successively without complication. Using the regular perturbation expansions for $A_\epsilon(\tau)$ and for $U(A_\epsilon(\tau), t, \epsilon)$ in the ansatz (19) results in an approximation for x_ϵ that agrees asymptotically with the naive expansion on intervals where t is finite, and that extends that approximation to longer intervals, at least as long as τ remains finite and $A_0(\tau)$ is defined. One possible further difficulty is instability of $A_0(\tau)$ as $\tau \to \infty$ (if $x(0)$ isn't restricted to the appropriate stable manifold). Another is encountering τ-secular terms in the power series generated for $A_\epsilon(\tau)$. Note, indeed, that A_1 will be τ-secular if the forcing term $a_1(A_0)$ contains a nontrivial projection in the range of the fundamental matrix $\frac{\partial A_0(\tau)}{\partial x(0)}$. A bounded τ, indeed, provides the usual time limit for obtaining asymptotic solutions by the classical averaging and two-timing methods, which are quite intimately related (cf. Morrison (1966)). Instability as $\tau \to \infty$ cannot be overcome. If, however, $A_0(\tau)$ exists for all $\tau \geq 0$ and if it decays exponentially to an asymptotically stable rest point or *sink*, the resulting expansion (24) for $A_\epsilon(\tau)$ and the resulting expansion (19) for $x_\epsilon(t)$ are uniformly valid for all $t \geq 0$. Recall that Greenlee and Snow (1975) provided an early discussion of such problems, while Murdock and Wang (1996) called this the Sanchez–Palencia condition, in reference to related results for averaging. Indeed, when $a_0(A_0) \equiv 0$, we can immediately seek the asymptotic solution $x_\epsilon(t)$ on $\mathcal{O}(1/\epsilon^2)$ time intervals, as Sanders and Verhulst (1985) and Murdock and Wang (1996) show for averaging and multiple scales, respectively, by replacing the slow-time τ in (21) by the even slower-time

$$\kappa = \epsilon\tau = \epsilon^2 t. \tag{27}$$

Readers should realize that the successful ansatz (19) can be interpreted as a *near-identity* transformation for z_ϵ. Such transformations, which generalize a classical asymptotic procedure of von Ziepel, were introduced by Neu (1980). They are useful in a variety of contexts, including many where our periodicity assumption doesn't hold. In this sense, the basic method of matched asymptotic expansions (cf. Il'in (1992)) and the boundary function method of Vasil'eva, Butuzov, and Kalachev (1995) can both be considered to be extensions of our renormalization technique, as will be demonstrated below. Note further that the basic ansatz (19) is considerably more direct than traditional two-timing, since the solution is not sought as an arbitrary function of the slow-time τ, but rather as a function of t and the amplitude A_ϵ, which is obtained asymptotically as a function of τ by solving the renormalized equation (21). At any stage, we have available a finite truncation

$$U^n(A_\epsilon(\tau), t, \epsilon) \equiv \sum_{j=0}^{n} \epsilon^j U_j(A_\epsilon(\tau), t)$$

for the correction U to A_ϵ satisfying $U(A_\epsilon(\tau), t, \epsilon) = U^n(A_\epsilon(\tau), t, \epsilon) + \mathcal{O}(\epsilon^{n+1})$. Likewise, we have the truncation

$$A_\epsilon^n(\tau) = \sum_{j=0}^{n} \epsilon^j A_j(\tau)$$

such that $A_\epsilon(\tau) = A_\epsilon^n(\tau) + \mathcal{O}(\epsilon^{n+1})$. Substituting into the integral (22), this implies that

$$U(A_\epsilon(\tau), t, \epsilon) = U^n(A_\epsilon^m(\tau), t, \epsilon) + \mathcal{O}(\epsilon^{n+1}) + \mathcal{O}(\epsilon^{m+2} t) \tag{28}$$

for any positive integers m and n, at least for appropriate bounded values of τ.

Our success in using the ansatz (19) for large times suggests that we might often be able to asymptotically solve the amplitude equation (21) on long time intervals, even when τ-secular terms in the series (24) for the amplitude A_ϵ need to be eliminated, by using a secondary ansatz

$$x(0) = B_\epsilon(\kappa) + \epsilon W(B_\epsilon(\kappa), \tau, \epsilon), \tag{29}$$

analogous to (19), in (19). We can asymptotically solve the resulting second level RG equation for the amplitude $B_\epsilon(\kappa)$ to get the resulting multiscale composite expansion

$$x_\epsilon(t) = e^{Mt}[\mathcal{A}(\tau, B_\epsilon(\kappa) + \epsilon W(B_\epsilon(\kappa), \tau, \epsilon), \epsilon) \\ + \epsilon U(\mathcal{A}(\tau, B_\epsilon(\kappa) + \epsilon W(B_\epsilon(\kappa), \tau, \epsilon), \epsilon), t, \epsilon)],$$

where we have set

$$A_\epsilon(\tau) = \mathcal{A}(\tau, x(0), \epsilon)$$

to emphasize its dependence on the initial vector $x(0)$. This expansion can be expected to be valid at least for bounded κ intervals. Moreover, we can consider the preceding expansion (19) to be an *intermediate* asymptotic expansion in the sense of Barenblatt (1996).

The critical idea behind the traditional (first level) renormalization group method of Chen, Goldenfeld, and Oono (1996) is to replace the initial value $x(0)$ in the naive expansion (7) by a slowly-varying function $A_\epsilon(\tau)$ through a near-identity transformation

$$x(0) = A_\epsilon(\tau) + \epsilon Z(A_\epsilon(\tau), t, \epsilon) \tag{30}$$

to eliminate secular (or *divergent*) terms in the naive expansion (7) by appropriate selection of the correction terms Z_j and to thereby obtain the secular-free expansion (19), where A_ϵ remains to be determined. To lowest orders, we would, for example, obtain the necessary cancellation by taking

$$Z_0(A_\epsilon, t) = -a_0(A_\epsilon)t$$

and

$$Z_1(A_\epsilon, t) = \frac{1}{2}t^2\langle f_x(A_\epsilon, t, 0)\rangle a_0(A_\epsilon) \\ - t\left[\langle f_x(A_\epsilon, t, 0)U_0(A_\epsilon, t, 0) + f_\epsilon(A_\epsilon, t, 0)\rangle - a_0(A_\epsilon)\left\langle \frac{\partial U_0}{\partial x}(A_\epsilon, t)\right\rangle\right].$$

Upon differentiating (30) with respect to t, the *invariance condition* $\frac{dx(0)}{dt} = 0$ and the chain rule immediately imply that $A_\epsilon(\tau)$ will satisfy

$$\frac{dA_\epsilon}{d\tau} = a(A_\epsilon, \epsilon) \equiv -\left(I + \epsilon\frac{\partial Z}{\partial A_\epsilon}\right)^{-1}\frac{\partial Z}{\partial t}. \tag{31}$$

We did not take this approach above because it is more direct to immediately find A_ϵ by asymptotically integrating (21), which turns out to ultimately be independent of the secular correction Z introduced in (30). We nonetheless note how instructive it is to see how the terms of the t-secular function Z can be selected to eliminate successive secular terms in (7) and to learn how the function $a(A_\epsilon, \epsilon)$, generated by using the intermediate Z, coincides with that already defined in (21). In some sense, renormalization corresponds to a summing of secular terms. We note that Nozaki and Oono (2001) get the RG equation from an intermediate proto-RG equation and that they make a distinction between resonant and nonresonant secular terms. Indeed, when no secular terms occur in (7), A_ϵ will remain the constant $x(0)$. Next, τ-secular terms in the resulting series (24) could analogously also be eliminated, if necessary, by replacing the initial vector $x(0)$ by a slowly varying function $B_\epsilon(\kappa)$ of the slow time $\kappa = \epsilon^2 t$ through use of another near-identity transformation (29), where $B_\epsilon(\kappa)$ must satisfy a second level RG equation

$$\frac{dB_\epsilon}{d\kappa} = b(B_\epsilon, \epsilon) \equiv -\left(I + \epsilon \frac{\partial W}{\partial B_\epsilon}\right)^{-1} \frac{\partial W}{\partial \tau} \tag{32}$$

and $B_\epsilon(0) = x(0)$ (cf. Mudavanhu and O'Malley (2001)). Assuming existence and appropriate stability of $B_0(\kappa)$, this will allow the asymptotic solution for x_ϵ to be defined beyond bounded values of κ. One may again be stopped by either blowup at finite κ, instability as $\kappa \to \infty$, or by κ-secular terms. The latter would require a higher level renormalization, and that could determine the asymptotic solution on a still longer time interval. We thus proceed in a leapfrog fashion. (Related applied work is contained in Moise and Ziane (2001) and Wirosoetisno, Shepherd, and Temam (2002).)

Two simple scalar examples. (a) Consider the simple example

$$\dot{x} = ix + \epsilon x(\alpha + x) \tag{33}$$

for some bounded complex constant α. Direct integration of this Riccati equation provides the exact solution

$$x_\epsilon(t) = e^{(i+\epsilon\alpha)t}\left[1 - \frac{\epsilon x(0)}{i+\epsilon\alpha}(e^{(i+\epsilon\alpha)t} - 1)\right]^{-1} x(0) \tag{34}$$

with a least period $\frac{2\pi}{1-i\epsilon\alpha}$ when $\operatorname{Re}\alpha = 0$. When $\alpha \neq 0$, secular terms become apparent when $e^{\epsilon\alpha t}$ is expanded in its Maclaurin series about $\epsilon = 0$. When $\operatorname{Re}\alpha < 0$, such a naive expansion in powers of ϵ is very misleading, since the actual solution decays exponentially to zero as $\tau = \epsilon t \to \infty$, while the Taylor-expanded series has unbounded secular terms. When $\operatorname{Re}\alpha > 0$, however, the solution blows up algebraically like $-\frac{i}{\epsilon}$ as $\tau \to \infty$. Thus, we can't expect an asymptotic approximation to the solution to be defined on time intervals on which τ becomes unbounded.

If we directly seek a solution to (33) of the form

$$x_\epsilon(t) = e^{it}(x(0) + \epsilon u(t, x(0), \epsilon)), \tag{35}$$

the scaled correction u must satisfy the nonlinear equation

$$\begin{aligned} \dot{u} &= f(x(0) + \epsilon u, t, 0) \\ &\equiv (\alpha + x(0)e^{it})x(0) + \epsilon(\alpha + 2x(0)e^{it})u + \epsilon^2 e^{it} u^2 \end{aligned} \tag{36}$$

and $u(0, x(0), \epsilon) = 0$. The resulting regular perturbation series is determined termwise through the successive linear initial value problems

$$\begin{aligned} \dot{u}_0 &= (\alpha + x(0)e^{it})x(0), \quad u_0(0) = 0, \\ \dot{u}_1 &= (\alpha + 2x(0)e^{it})u_0, \quad u_1(0) = 0, \\ \dot{u}_2 &= (\alpha + 2x(0)e^{it})u_1 + e^{it}u_0^2, \quad u_2(0) = 0, \end{aligned}$$

etc. Integrating termwise, we obtain the naive expansion

$$
\begin{aligned}
x_\epsilon(t) = e^{it}\Bigg[& x(0) + \epsilon(\alpha t + ix(0)(1-e^{it}))x(0) \\
& + \epsilon^2 x(0)\bigg[\frac{1}{2}\alpha^2 t^2 + i\alpha x(0)t(1-2e^{it}) \\
& - x(0)(1-e^{it})(\alpha + x(0)(1-e^{it}))\bigg] + \mathcal{O}(\epsilon^3)\Bigg],
\end{aligned}
\tag{37}
$$

valid asymptotically for bounded t values. The anticipated secular terms occur, however, for $\alpha \neq 0$, indicating the breakdown of the approximation (37) when $\tau = \epsilon t \to \infty$.

If we instead seek an asymptotic solution $x_\epsilon(t)$ of (33), using our ansatz

$$x_\epsilon(t) = e^{it}(A_\epsilon(\tau) + \epsilon U(A_\epsilon(\tau), t, \epsilon)), \tag{38}$$

the amplitude A_ϵ and the correction U will have to satisfy (21) and (22), respectively. Since $f(A_\epsilon(\tau), t, 0) = (\alpha + A_\epsilon(\tau)e^{it})A_\epsilon(\tau)$, the average

$$\langle f(A_\epsilon(\tau), t, 0)\rangle = \alpha A_\epsilon(\tau)$$

is the leading term of its Fourier series, and it is supplemented by

$$\{f(A_\epsilon(\tau), t, 0)\} = A_\epsilon^2(\tau)e^{it}.$$

This implies that both

$$\frac{dA_\epsilon}{d\tau} = \alpha A_\epsilon + \mathcal{O}(\epsilon)$$

and

$$U_0(A_\epsilon(\tau), t) = iA_\epsilon^2(\tau)(1 - e^{it}),$$

and thus

$$A_\epsilon(\tau) = e^{\alpha\tau}x(0) + \mathcal{O}(\epsilon)$$

is defined on all $\tau \geq 0$, provided $\operatorname{Re}\alpha \leq 0$. Otherwise, the solution $x_\epsilon(t)$ will be bounded only for finite τ.

The next-order corrections to $\frac{dA_0}{d\tau}$ and U_0 involve the expression

$$
\begin{aligned}
& f_x(A_\epsilon, t, 0)U_0(A_\epsilon, t) - \frac{\partial U_0}{\partial A_\epsilon}(A_\epsilon, t)\langle f(A_\epsilon, t, 0)\rangle \\
& \quad = (-i\alpha A_\epsilon^2 + 2iA_\epsilon^3 e^{it})(1 - e^{it}).
\end{aligned}
$$

Since its average part is $-i\alpha A_\epsilon^2$, the $\mathcal{O}(\epsilon)$ improved approximations satisfy the amplitude equation

$$\frac{dA_\epsilon}{d\tau} = \alpha A_\epsilon - \epsilon i\alpha A_\epsilon^2 + \mathcal{O}(\epsilon^2), \tag{39}$$

and the corresponding secular-free correction to A_ϵ is given by

$$U(A_\epsilon(\tau), t, \epsilon) = iA_\epsilon^2(1 - e^{it}) - \epsilon iA_\epsilon^2(1 - e^{it})(\alpha + A_\epsilon(1 - e^{it})) + \mathcal{O}(\epsilon^2). \tag{40}$$

As expected, the latter coincides with the secular-free terms of the appropriately truncated naive expansion, with the slowly varying amplitude A_ϵ replacing the initial value $x(0)$.

For $\operatorname{Re}\alpha < 0$, $A_0(\tau)$ will decay exponentially to zero, and this allows us to obtain the asymptotic solution (38) for all $\tau \geq 0$. The most interesting case occurs when $\operatorname{Re}\alpha = 0$. Then, the two-term truncation of the amplitude equation (39) is essentially the same as the original Riccati equation (33). To find the solution for $\kappa = \epsilon^2 t = \mathcal{O}(1)$ generally requires another renormalization, so we will then obtain an asymptotic representation of the solution in terms of the three scales, t, τ, and an amplitude B_ϵ that is a function of κ.

A numerical verification of any presumed approximation X can be carried out by employing the technique of Bosley (1996). We define the absolute error

$$E(t,\epsilon) = |x_{\text{exact}} - X|,$$

although we often employ a carefully computed numerical solution in place of the unknown exact solution. If $E = \mathcal{O}(\epsilon^{n+1})$ for a fixed time, the value of $\log(E)$ as a function of $\log\epsilon$ should be linear with slope $n+1$ in the limit $\epsilon \to 0$. The slope is readily determined by using a linear least squares fit. For the example, interesting results are obtained by considering the three separate cases: $\alpha = 1$, $\alpha = -1$, and $\alpha = i$. (Comparable conclusions on longer time intervals naturally follow for the example

$$\dot{x} = i\epsilon x + \epsilon^2 x(\alpha + x),$$

which we treat by immediately introducing $\tau = \epsilon t$ as a replacement for the given time t.)

Figures 1–3 below are comparisons of the exact solutions in these three cases with, respectively, regular perturbation and RG asymptotic approximations for (33), together with their numerical verifications of asymptotic validity using Bosley's technique for $t = 10$.

(b) We next consider the nonautonomous equation

$$\dot{x} = \epsilon N(x,t) \equiv \epsilon(-x^3 - x^2\cos t + \sin t), \tag{41}$$

introduced by Murdock and Wang (1996), together with a positive initial value $x(0)$. Since $x\dot{x} < 0$ for $|x|$ sufficiently large, the solution $x_\epsilon(t)$ remains bounded for all times. Setting

$$x_\epsilon(t) = x(0) + \epsilon u(t, x(0), \epsilon),$$

it follows that u must satisfy the initial value problem

$$\begin{aligned}\dot{u} = (-x^3(0) + x^2(0)\cos t + \sin t) + \epsilon(-3x^2(0) + 2x(0)\cos t)u \\ + \epsilon^2(-3x(0) + \cos t)u^2, \quad u(0) = 0,\end{aligned}$$

for which a naive expansion could be readily generated. Alternatively, the ansatz (or near-identity transformation)

$$x_\epsilon(t) = A_\epsilon(\tau) + \epsilon U(A_\epsilon(\tau), t, \epsilon) \tag{42}$$

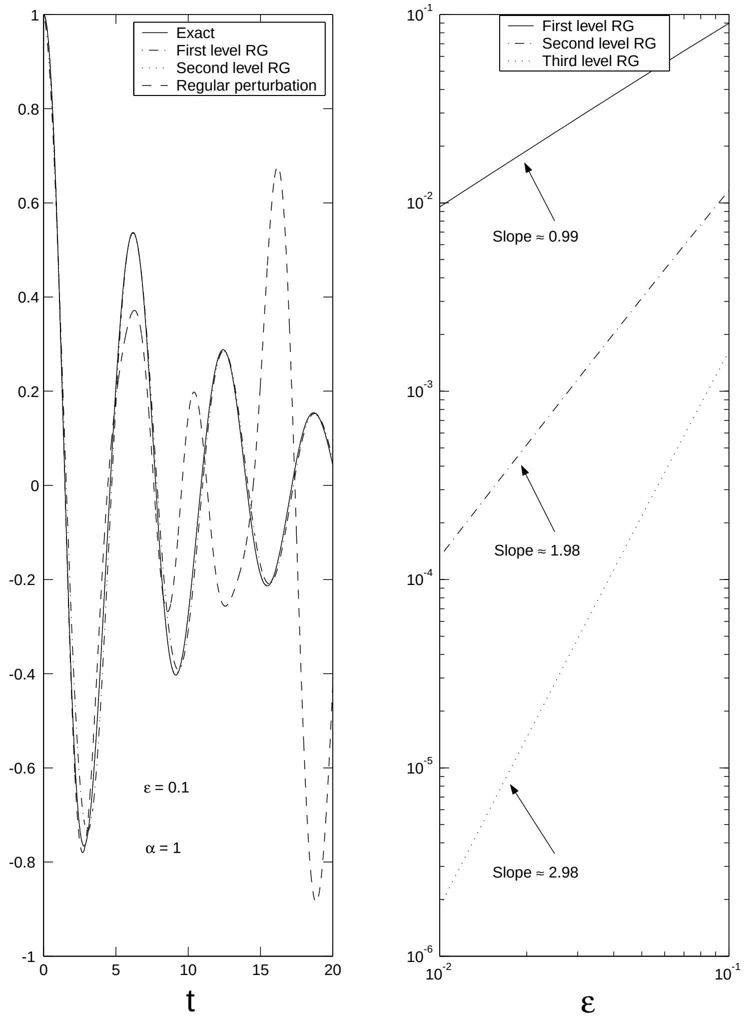

FIG. 1.

involves, to leading order, the amplitude equation

$$\frac{dA_\epsilon}{d\tau} = \langle N(A_\epsilon, t)\rangle + \mathcal{O}(\epsilon) = -A_\epsilon^3 + \mathcal{O}(\epsilon),$$

predicted by averaging, and the secular-free correction term

$$U_0(A_\epsilon, t) = \int_0^t \{N(A_\epsilon, s)\}ds = A_\epsilon^2(\tau)\sin t + 1 - \cos t.$$

Since the resulting limiting amplitude

$$A_0(\tau) = \frac{x(0)}{\sqrt{2\tau x^2(0) + 1}} \tag{43}$$

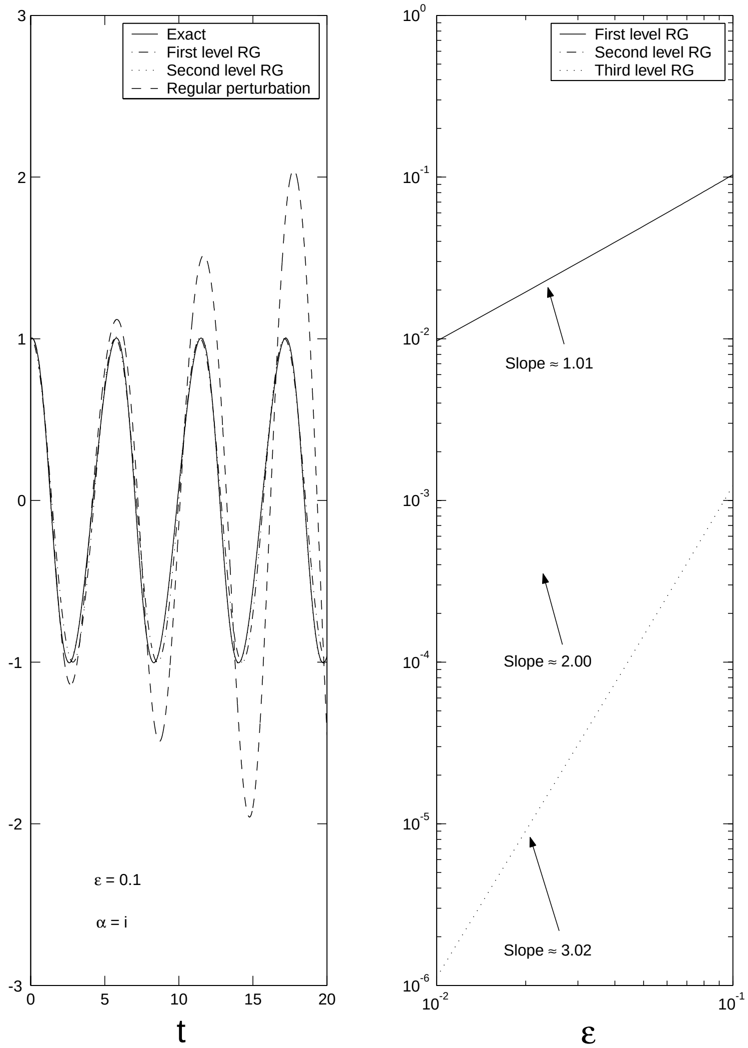

FIG. 2.

decays only algebraically as $\tau \to \infty$, we naturally seek its $\mathcal{O}(\epsilon)$ correction determined by using the average part of the expression

$$\begin{aligned} &N_x(A_0,t)U_0(A_0,t) - \langle N(A_0,t)\rangle U_{0x}(A_0,t) \\ &\quad = -(3A_0^2 + A_0) - A_0^4 \sin t + (3A_0^2 + 2A_0)\cos t + A_0^3 \sin 2t - A_0 \cos 2t. \end{aligned}$$

Since this implies the more accurate amplitude equation

$$\frac{dA_\epsilon}{d\tau} = -A_\epsilon^3 - \epsilon(3A_\epsilon^2 + A_\epsilon) + \mathcal{O}(\epsilon^2), \tag{44}$$

the regular perturbation series

$$A_\epsilon(\tau) = A_0(\tau) + \epsilon A_1(\tau) + \mathcal{O}(\epsilon^2) \tag{45}$$

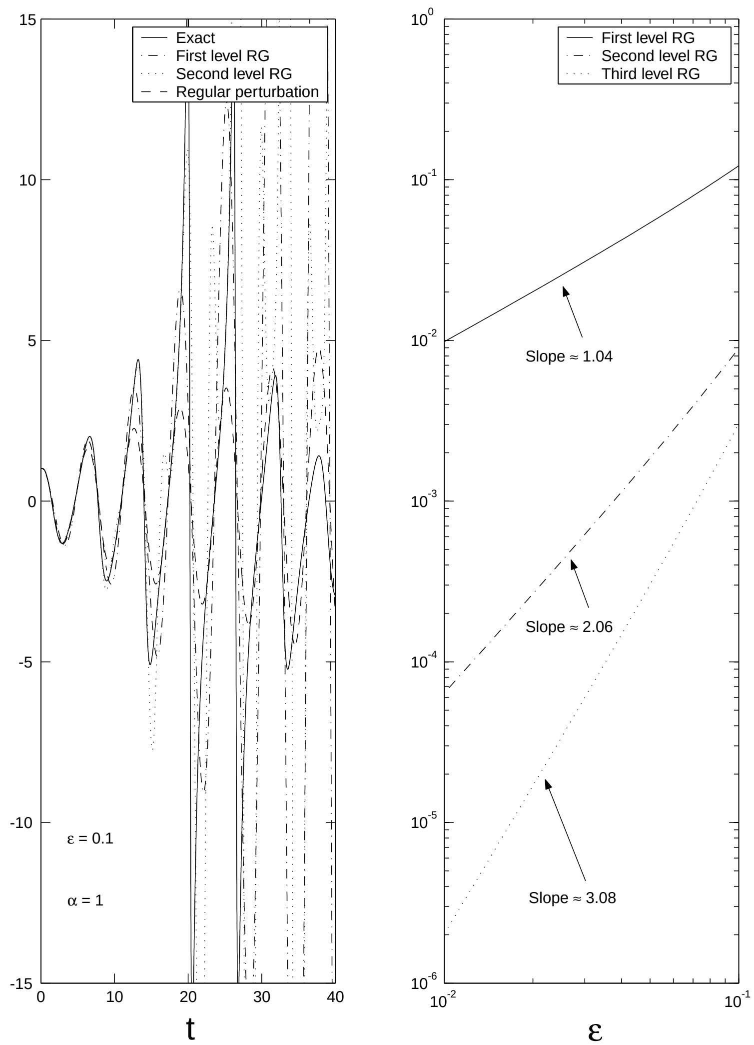

FIG. 3.

will require that the first correction term A_1 satisfy the linear initial value problem

$$\frac{dA_1}{d\tau} = -3A_0^2 A_1 - (3A_0^2 + A_0), \quad A_1(0) = 0.$$

We write its exact solution as

$$A_1(\tau) = -\frac{\sqrt{2\tau x^2(0) + 1}}{4x(0)} - 1 + \left(\frac{1}{4x(0)} + 1\right)\frac{1}{(\sqrt{2\tau x^2(0) + 1})^3}. \tag{46}$$

Since $|A_1|$ blows up like $\tau^{1/2}$ as $\tau \to \infty$, we shall attempt to eliminate its secular term and later ones in (45) by using a traditional renormalization. Setting

$$x(0) = B_\epsilon(\kappa) + \epsilon W(B_\epsilon(\kappa), \tau, \epsilon) \tag{47}$$

in (45) and using a power series for W, we get the power series expansion

$$
\begin{aligned}
A_\epsilon(\tau) &= \frac{B_\epsilon + \epsilon W}{\sqrt{2\tau(B_\epsilon + \epsilon W)^2 + 1}} - \epsilon\left[\frac{\sqrt{2\tau(B_\epsilon + \epsilon W)^2 + 1}}{4(B_\epsilon + \epsilon W)} - 1 + \cdots\right] + \cdots \\
&= \frac{B_\epsilon}{\sqrt{2\tau B_\epsilon^2 + 1}} + \epsilon\left[\frac{W_0}{\sqrt{2\tau B_\epsilon^2 + 1}} - \frac{2\tau B_\epsilon^2 W_0}{(\sqrt{2\tau B_\epsilon^2 + 1})^3} - \frac{\sqrt{2\tau B_\epsilon^2 + 1}}{4B_\epsilon} + \cdots\right] + \cdots .
\end{aligned}
$$

Thus, we can cancel the troublesome τ-secular term at $\mathcal{O}(\epsilon)$ by picking

$$W_0(B_\epsilon, \tau) = \frac{(2\tau B_\epsilon^2 + 1)^2}{4B_\epsilon}.$$

The resulting second level RG equation (32) is

$$\frac{dB_\epsilon}{d\kappa} = -\frac{\partial W_0}{\partial \kappa} + \mathcal{O}(\epsilon) = -\frac{2}{\epsilon}\kappa B_\epsilon^2 - B_\epsilon + \mathcal{O}(\epsilon).$$

Solving the two-term approximate Riccati equation with the initial value $B_\epsilon(0) = x(0)$ determines the exponentially decaying

$$B_0(\kappa) = \frac{e^{-\kappa}x(0)}{\sqrt{\frac{x^2(0)}{\epsilon}(1 - e^{-2\kappa} - 2\kappa e^{-2\kappa}) + 1}} \tag{48}$$

(with admitted abuse of notation) and the corresponding leading-order approximation

$$x_\epsilon(t) = \frac{B_0(\kappa)}{\sqrt{\frac{2\kappa}{\epsilon}B_0^2(\kappa) + 1}} + \mathcal{O}(\epsilon) \tag{49}$$

to the decaying solution, which is asymptotically valid for all $t \geq 0$. We note that the regular perturbation expansion is asymptotically correct for t finite, that the series (42) and (45) with τ-secular terms is likewise correct for τ finite, but that the twice-renormalized expansion corresponding to (49) is needed on longer time intervals. The algebraic decay of the limiting solution with $\sqrt{\kappa/\epsilon} = \sqrt{\epsilon t}$ is unexpected, but it follows from renormalization, as does the ultimate exponential decay as $\kappa \to \infty$. Analogous behavior was obtained in Mudavanhu and O'Malley (2001) in solving the second-order equation

$$\ddot{y} + y + \epsilon\dot{y}^3 + 3\epsilon^2\dot{y} = 0,$$

introduced in Morrison (1966).

Figure 4 is a comparison of the numerical solution and the first and second *level* RG asymptotic approximations for the solution of (41). The second level approximation is obtained by renormalizing the second-order amplitude equation as indicated. Figure 5 shows the numerical verifications of the RG approximations using Bosley's technique for $t = 10$.

Second-order scalar equations. Mudavanhu and O'Malley (2001) considered scalar equations of the form

$$\ddot{y} + y + \epsilon g(y, \dot{y}, \epsilon) = 0 \tag{50}$$

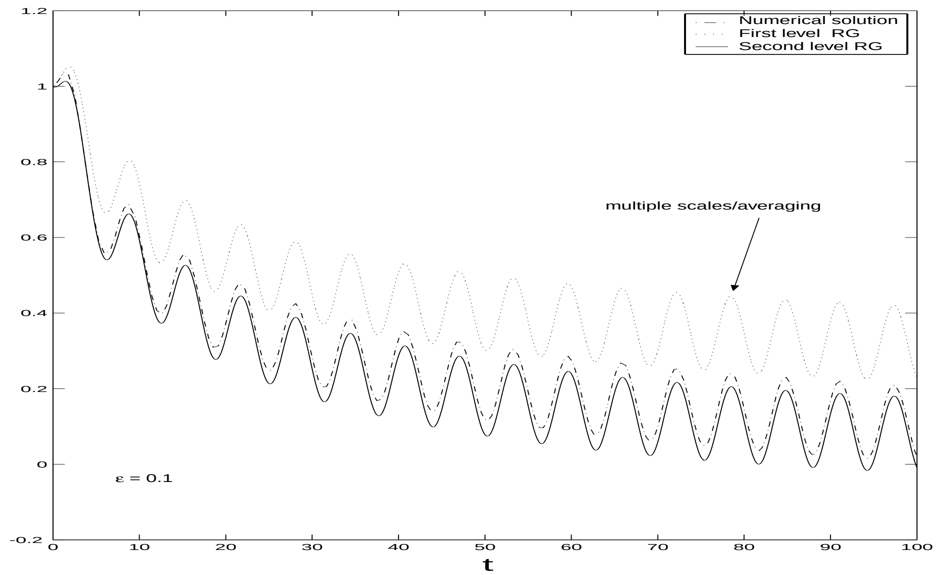

Fig. 4.

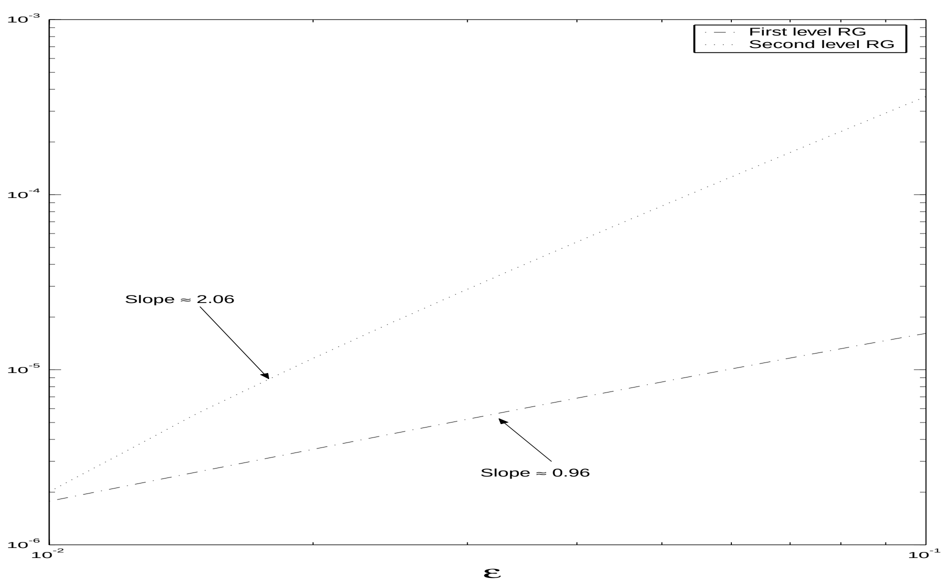

Fig. 5.

on $t \geq 0$, with $y(0)$ and $\dot{y}(0)$ prescribed. Such problems take the form (1) when one introduces

$$x = \begin{pmatrix} y \\ \dot{y} \end{pmatrix}, \qquad M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \text{and } N(x,\epsilon) = \begin{pmatrix} 0 \\ -g(y,\dot{y},\epsilon) \end{pmatrix}, \tag{51}$$

and thus their asymptotic solution on appropriate time intervals is determined by the preceding.

It is more traditional, however, to use the Prüfer transformation

$$y = \rho_\epsilon(t)\cos(t+\psi_\epsilon(t)), \qquad \dot{y} = -\rho_\epsilon(t)\sin(t+\psi_\epsilon(t)) \tag{52}$$

to obtain a system of differential equations for the nonnegative amplitude ρ_ϵ and the phase ψ_ϵ. As in variation of parameters,

$$\dot{y} = \frac{d\rho_\epsilon}{dt}\cos(t+\psi_\epsilon) - \rho_\epsilon\sin(t+\psi_\epsilon)\left(1+\frac{d\psi_\epsilon}{dt}\right)$$

and

$$\ddot{y} = -\frac{d\rho_\epsilon}{dt}\sin(t+\psi_\epsilon) - \rho_\epsilon\cos(t+\psi_\epsilon)\left(1+\frac{d\psi_\epsilon}{dt}\right)$$

imply a linear algebraic system for $\frac{d\rho_\epsilon}{dt}$ and $\frac{d\psi_\epsilon}{dt}$ that yields

$$\begin{cases} \dfrac{d\rho_\epsilon}{dt} = \epsilon\sin(t+\psi_\epsilon)g(\rho_\epsilon\cos(t+\psi_\epsilon), -\rho_\epsilon\sin(t+\psi_\epsilon)), \\ \dfrac{d\psi_\epsilon}{dt} = \dfrac{\epsilon}{\rho_\epsilon}\cos(t+\psi_\epsilon)g(\rho_\epsilon\cos(t+\psi_\epsilon), -\rho_\epsilon\sin(t+\psi_\epsilon)). \end{cases} \tag{53}$$

The needed initial values $\rho_\epsilon(0)$ and $\psi_\epsilon(0)$ for (53) are likewise uniquely specified since

$$y(0) = \rho_\epsilon(0)\cos\psi_\epsilon(0) \text{ and } \dot{y}(0) = -\rho_\epsilon(0)\sin\psi_\epsilon(0). \tag{54}$$

Since ψ_ϵ occurs in the combination $z \equiv t+\psi_\epsilon(t)$, we can rewrite (53) as a 2π-periodic function of z:

$$\begin{cases} \dfrac{d\rho_\epsilon}{dz} = \dfrac{\epsilon\rho_\epsilon\sin z\, g(\rho_\epsilon\cos z, -\rho_\epsilon\sin z)}{\rho_\epsilon + \epsilon\cos z\, g(\rho_\epsilon\cos z, -\rho_\epsilon\sin z)}, \\ \dfrac{d\psi_\epsilon}{dz} = \dfrac{\epsilon\cos z\, g(\rho_\epsilon\cos z, -\rho_\epsilon\sin z)}{\rho_\epsilon + \epsilon\cos z\, g(\rho_\epsilon\cos z, -\rho_\epsilon\sin z)}. \end{cases} \tag{55}$$

Our ansatz (19) suggests seeking an asymptotic solution for (55) in the form

$$\begin{cases} \rho_\epsilon(t) = R_\epsilon(\tau) + \epsilon U(R_\epsilon(\tau), \Psi_\epsilon(\tau), t, \epsilon), \\ \psi_\epsilon(t) = \Psi_\epsilon(\tau) + \epsilon V(R_\epsilon(\tau), \Psi_\epsilon(\tau), t, \epsilon). \end{cases} \tag{56}$$

The advantage obtained is that the first-order renormalized system is triangular, i.e.,

$$\begin{cases} \dfrac{dR_\epsilon}{d\tau} = \alpha(R_\epsilon, \epsilon) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \sin z\, g(R_\epsilon\cos z, -R_\epsilon\sin z)dz + \mathcal{O}(\epsilon), \\ \dfrac{d\Psi_\epsilon}{d\tau} = \beta(R_\epsilon, \epsilon) = \dfrac{1}{2\pi R_\epsilon}\displaystyle\int_0^{2\pi} \cos z\, g(R_\epsilon\cos z, -R_\epsilon\sin z)dz + \mathcal{O}(\epsilon). \end{cases} \tag{57}$$

Note that $\alpha(R_\epsilon, 0)$ and $\beta(R_\epsilon, 0)$ are half of the corresponding first harmonic coefficients in the Fourier series for $g(R_\epsilon\cos z, -R_\epsilon\sin z)$ on $0 \le z \le 2\pi$. It's an easy system to solve implicitly as

$$\tau = \int_{R_\epsilon(0)}^{R_\epsilon} \frac{du}{\alpha(u,\epsilon)} \quad \text{and} \quad \Psi_\epsilon(\tau) = \psi_\epsilon(0) + \int_0^\tau \beta(R_\epsilon(p), \epsilon)dp, \tag{58}$$

although specifying where $R_\epsilon(\tau)$ is well defined involves all the anticipated complications. By the chain rule, it also follows that

(59)
$$\begin{cases} U_0(R_\epsilon(\tau),\Psi_\epsilon(\tau),t) = \displaystyle\int_0^t [\sin(s+\Psi_\epsilon(\tau))g(R_\epsilon(\tau)\cos(s+\Psi_\epsilon(\tau)) \\ \qquad\qquad - R_\epsilon(\tau)\sin(s+\Psi_\epsilon(\tau))) - \alpha(R_\epsilon(\tau),0)]ds \\ \text{and} \\ V_0(R_\epsilon(\tau),\Psi_\epsilon(\tau),t) = \dfrac{1}{R_\epsilon(\tau)}\displaystyle\int_0^t [\cos(s+\Psi_\epsilon(\tau))g(R_\epsilon(\tau)\cos(s+\Psi_\epsilon(\tau)) \\ \qquad\qquad - R_\epsilon(\tau)\sin(s+\Psi_\epsilon(\tau))) - \beta(R_\epsilon(\tau),0)]ds. \end{cases}$$

Moreover, using the ansatz (56), we get the secular-free approximations

(60)
$$\begin{cases} y = \rho\cos(t+\psi) = R_\epsilon\cos(t+\Psi_\epsilon) + \epsilon(U_0\cos(t+\Psi_\epsilon) - R_\epsilon V_0\sin(t+\Psi_\epsilon)) + \mathcal{O}(\epsilon^2) \\ \text{and} \\ \dot{y} = -\rho\sin(t+\psi) = -R_\epsilon\sin(t+\Psi_\epsilon) - \epsilon(U_0\sin(t+\Psi_\epsilon) + R_\epsilon V_0\cos(t+\Psi_\epsilon)) + \mathcal{O}(\epsilon^2). \end{cases}$$

Higher-order approximations follow without difficulty, even for many problems where classical methods break down.

(a) As a first concrete example, consider the Duffing–van der Pol equation

$$\ddot{y} + y + \epsilon y^3 + \epsilon^2(y^2-1)\dot{y} = 0, \tag{61}$$

introduced by Benney and Newell (1967). Seeking a solution as

$$y = \rho_\epsilon\cos(t+\psi_\epsilon) \quad \text{and} \quad \dot{y} = -\rho_\epsilon\sin(t+\psi_\epsilon)$$

provides the periodic forcing $g = y^3 + \epsilon(y^2-1)\dot{y}$ as

$$\begin{aligned} g(y,\dot{y},\epsilon) = {} & \frac{1}{4}\rho_\epsilon^3[3\cos(t+\psi_\epsilon) + \cos 3(t+\psi_\epsilon)] \\ & + \epsilon\bigg[-\frac{\rho_\epsilon^3}{4}(\sin(t+\psi_\epsilon) + \sin 3(t+\psi_\epsilon)) \\ & \qquad + \rho_\epsilon^2\sin 2(t+\psi_\epsilon) - \rho_\epsilon\sin(t+\psi_\epsilon)\bigg]. \end{aligned}$$

Since its leading term has a trivial $\sin z$ coefficient in its Fourier series and $\frac{3}{4}\rho_\epsilon^3$ as the $\cos z$ coefficient, we will have the amplitude and phase equations

$$\frac{d\rho_\epsilon}{d\tau} = \mathcal{O}(\epsilon) \quad \text{and} \quad \frac{d\psi_\epsilon}{d\tau} = \frac{3}{8}\rho_\epsilon^2 + \mathcal{O}(\epsilon).$$

(Note that Cox and Roberts (1995) and Roberts (1997) attain such reductions efficiently by *normal form* transformations implemented using REDUCE. Mudavanhu (2000) obtains the same results and corresponding higher-order terms via a renormalization method automated using MAPLE.) Indeed, our results suggest the more efficient introduction of the slower-time $\kappa = \epsilon^2 t$. Incorporating κ and using the next terms in (57), we obtain

$$\text{(62)} \qquad \begin{cases} \dfrac{dR_\epsilon}{d\kappa} = \dfrac{1}{2}R_\epsilon\left(1-\dfrac{1}{4}R_\epsilon^2\right)+\mathcal{O}(\epsilon) \\ \text{and} \\ \dfrac{d\Psi_\epsilon}{d\tau} = \dfrac{3}{8}R_\epsilon^2 - \dfrac{15\epsilon}{256}R_\epsilon^4+\mathcal{O}(\epsilon^2). \end{cases}$$

Solving the limiting Bernoulli equation determines an expansion for

$$\text{(63)} \qquad R_\epsilon(\kappa) = R_0(\kappa)+\epsilon R_1(\kappa)+\epsilon^2(\cdots)$$

as an exponentially decaying amplitude for all $\kappa \geq 0$, with leading term

$$R_0(\kappa) = \frac{2}{\sqrt{1-(1-4/\rho_\epsilon^2(0))e^{-\kappa}}}.$$

The resulting limit cycle behavior follows with the phase

$$\text{(64)} \qquad \Psi_\epsilon(t) = \psi_\epsilon(0)+\frac{3}{8}\int_0^t R_\epsilon^2(\epsilon^2 s)\left(1-\frac{5\epsilon}{32}R_\epsilon^2(\epsilon^2 s)\right)ds+\mathcal{O}(\epsilon^2 t).$$

In the integrand, it is clearly preferable to represent $R_\epsilon(\kappa)$ as the sum of its steady-state limit plus an exponentially decaying transient. Higher-order approximations to the solution follow as in (60).

Alternatively, we can use the transformation (51) and the spectral decomposition $M = iV\Lambda V^{-1}$ for a nonsingular modal matrix $V = \left(\begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix}\right)$ and $\Lambda = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Directly applying our basic ansatz,

$$\text{(65)} \qquad x_\epsilon(t) = Ve^{i\Lambda t}V^{-1}(A_\epsilon+\epsilon U(A_\epsilon(\tau),\epsilon)),$$

where $A_\epsilon = \left(\begin{smallmatrix} a_\epsilon \\ \bar{a}_\epsilon \end{smallmatrix}\right)$, for complex conjugates a_ϵ and $\bar{a}_\epsilon$, involves, to leading order, the amplitude equation

$$\frac{dA_\epsilon}{d\tau} = \langle e^{-i\Lambda t}V^{-1}N(Ve^{i\Lambda t}x,0)\rangle+\mathcal{O}(\epsilon) = \frac{3}{2}i|a_\epsilon|^2\begin{pmatrix} a_\epsilon \\ -\bar{a}_\epsilon \end{pmatrix}+\mathcal{O}(\epsilon)$$

and the secular-free correction term

$$\begin{aligned} U_0(A_\epsilon,t) &= \int_0^t \{e^{-i\Lambda s}V^{-1}N(Ve^{i\Lambda s}x,0)\}ds \\ &= \frac{1}{4}\begin{pmatrix} a_\epsilon^3 e^{-2it}-3\bar{a}_\epsilon|a_\epsilon|^2e^{2it}+\frac{1}{2}\bar{a}_\epsilon^3e^{4it} \\ \bar{a}_\epsilon^3e^{2it}-3a_\epsilon|a_\epsilon|^2e^{-2it}+\frac{1}{2}a_\epsilon^3e^{-4it} \end{pmatrix}. \end{aligned}$$

Letting $a_\epsilon = \frac{R_\epsilon}{2}e^{-i\Psi_\epsilon}$ provides the amplitude and phase equations $\frac{dR_\epsilon}{d\tau} = \mathcal{O}(\epsilon)$, $\frac{d\Psi_\epsilon}{d\tau} = \frac{3}{8}R_\epsilon^2+\mathcal{O}(\epsilon)$ as before and the corresponding asymptotic approximation

$$\text{(66)} \qquad y = R_\epsilon\cos(t+\Psi_\epsilon)+\epsilon\frac{R_\epsilon}{16}\left[3\cos(t+\Psi_\epsilon)+\frac{1}{2}\cos 3(t+\Psi_\epsilon)\right]+\epsilon^2(\cdots).$$

Higher-order approximations follow in a straightforward fashion.

(b) We finally seek the RG equations resulting from two weakly coupled van der Pol oscillators

$$\ddot{y} + \Omega y + \epsilon(I - \mathcal{C}^2)\dot{y} = \epsilon\mathcal{B}(\alpha y + \beta\dot{y}), \tag{67}$$

where

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \qquad \Omega = \begin{pmatrix} 1 & 0 \\ 0 & 1+\Delta \end{pmatrix}, \qquad \mathcal{C} = \begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix}, \qquad \mathcal{B} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \tag{68}$$

and I is an identity matrix (cf. Reinhall and Storti (2000) and Low (2002)). Here α and β are coupling constants, and Δ is a detuning parameter related to the difference in the natural frequencies of the two oscillators. We now seek asymptotic solutions of the form

$$\begin{cases} y_1(t,\epsilon) = R_1(\tau,\epsilon)\cos(t + \Psi_1(\tau,\epsilon)) + \epsilon(\cdots), \\ y_1(t,\epsilon) = R_2(\tau,\epsilon)\cos(t + \Psi_2(\tau,\epsilon)) + \epsilon(\cdots) \end{cases} \tag{69}$$

for the slow time $\tau = \epsilon t$, where R_j and Ψ_j, for $j = 1$ and 2, represent the amplitude and phase modulations. The functions y_j are said to be *phase locked* when the difference

$$\phi_\epsilon = \Psi_{2\epsilon} - \Psi_{1\epsilon} \tag{70}$$

is a constant. When the oscillators are running at unequal frequencies (i.e., $\Delta \neq 0$), ϕ_ϵ will grow unbounded, defining a condition known as a *phase drift*. An intermediate situation exists when ϕ_ϵ varies periodically, a condition known as *phase entrainment*.

Applying our basic ansatz, by first transforming to a four-dimensional system of the form (1), we systematically obtain the RG equations

$$2\frac{dA_{1\epsilon}}{d\tau} = A_{1\epsilon}(1 - |A_{1\epsilon}|^2) - \beta(A_{1\epsilon} - A_{2\epsilon}) + i\alpha(A_{1\epsilon} - A_{2\epsilon}) + \epsilon(\cdots),$$
$$2\frac{dA_{2\epsilon}}{dt} = A_{2\epsilon}(1 - |A_{2\epsilon}|^2) - \beta(A_{2\epsilon} - A_{1\epsilon}) + i\alpha(A_{2\epsilon} - A_{1\epsilon}) + i\frac{\Delta}{2}A_{1\epsilon} + \epsilon(\cdots).$$

Letting $A_{j\epsilon} = R_{j\epsilon}e^{-i\Psi_{j\epsilon}}$ for $j = 1$ and 2, we get the system of three slowly varying RG equations

$$\begin{cases} 2\dfrac{dR_{1\epsilon}}{d\tau} = (1 - R_{1\epsilon}^2)R_{1\epsilon} - \beta(R_{1\epsilon} - R_{2\epsilon}\cos\phi_\epsilon) + \alpha R_{2\epsilon}\sin\phi_\epsilon + \epsilon(\cdots), \\ 2\dfrac{dR_{2\epsilon}}{d\tau} = (1 - R_{2\epsilon}^2)R_{2\epsilon} - \beta(R_{2\epsilon} - R_{1\epsilon}\cos\phi_\epsilon) \; - \alpha R_{1\epsilon}\sin\phi_\epsilon + \epsilon(\cdots), \\ 2\dfrac{d\phi_\epsilon}{d\tau} = \Delta - \beta\left(\dfrac{R_{2\epsilon}}{R_{1\epsilon}} - \dfrac{R_{1\epsilon}}{R_{2\epsilon}}\right)\sin\phi_\epsilon - \alpha\left(\dfrac{R_{2\epsilon}}{R_{1\epsilon}} + \dfrac{R_{1\epsilon}}{R_{2\epsilon}}\right)\cos\phi_\epsilon + \epsilon(\cdots). \end{cases} \tag{71}$$

Stability analyses of (67) can be carried out based on these and higher-order amplitude equations (cf. Chakraborty and Rand (1988)).

Relation to two-timing and other classical techniques. The asymptotic solution of the initial value problem (3)

$$\dot{z} = \epsilon f(z, t, \epsilon)$$

(in standard form) could be obtained using the two-time ansatz

$$z = z(t, \tau, \epsilon)$$

for the bounded slow-time $\tau = \epsilon t$. The chain rule implies that

$$\dot{z} = \frac{\partial z}{\partial t} + \epsilon \frac{\partial z}{\partial \tau},$$

and thus substituting a power series expansion

$$z(t, \tau, \epsilon) = z_0(t, \tau) + \epsilon z_1(t, \tau) + \cdots \tag{72}$$

into (3) requires that

$$\frac{\partial z_0}{\partial t} = 0 \tag{73}$$

and

$$\frac{\partial z_j}{\partial t} = g_{j-1}(t, z_0, z_1, \dots, z_{j-1}) - \frac{\partial z_{j-1}}{\partial \tau} \tag{74}$$

for each $j \geq 1$. Here

$$f(z(t, \tau, \epsilon), t, \epsilon) \sim \sum_{j=0}^{\infty} \epsilon^j g_j(t, z_0, \dots, z_{j-1}, z_j),$$

where

$$g_j(t, z_0, \dots, z_{j-1}, z_j) - f_z(z_0(t, \tau), t, 0) z_j$$

is a known function of the earlier coefficients $z_0, z_1, \dots, z_{j-1}$ and t.

We first obtain the representation

$$z_0(t, \tau) = A_0(\tau) \tag{75}$$

from integrating (73), for some unspecified $A_0(\tau)$. Taking $j = 1$, we then find that

$$\frac{\partial z_1}{\partial t} = f_0(A_0(\tau), t) - \frac{dA_0}{d\tau}. \tag{76}$$

Recall that f_0 is a periodic function of t. To get the boundedness of z_1 as $t \to \infty$ requires the right-hand side to have zero average, i.e., A_0 must be the unique solution of the initial value problem for

$$\frac{dA_0}{d\tau} = \langle f_0(A_0(\tau), t) \rangle. \tag{25}$$

This leaves $\frac{\partial z_1}{\partial t} = \{f_0(A_0(\tau), t)\}$, and so

$$z_1(t, \tau) = A_1(\tau) + U_0(A_0(\tau), t) \tag{77}$$

for an unknown A_1 and the bounded function $U_0 = \int_0^t \{f_0(A_0(\tau), s)\} ds$, already encountered. If we next consider (74) for $j = 2$, the boundedness of z_2 will require the average of the right-hand side to be zero. This, however, shows that A_1 must satisfy an initial value problem of the form

$$\frac{dA_1}{d\tau} = \left\langle \frac{\partial f_0}{\partial z}(A_0(\tau), t) \right\rangle A_1 + \widetilde{a_1}(A_0(\tau)),$$

with a known inhomogeneity $\widetilde{a}_1$ and the trivial initial value. The unique solution follows as in (26). Continuing, in this manner, to use the Fredholm alternative to get a bounded solution at every stage, we obtain our two-time expansion to any order.

Murdock and Wang (1996) prove that this result is asymptotically valid, to all orders, for finite τ. An attempt to comprehend renormalization has thus provided an opportunity to rethink two-timing. We point out, however, that a less restrictive method of *slowly varying amplitudes* is often used in applications (cf., e.g., the final chapter of Haberman (1998), in addition to the literature already cited). It compares to our ansatz (19),

$$z_\epsilon(t) = A_\epsilon(\tau) + \epsilon U(A_\epsilon(\tau), t, \epsilon),$$

rather than the more general two-timing expansion (72). The idea is to seek an amplitude $A_\epsilon(\tau)$, varying with the slow time τ and ϵ, so that secular terms in the resulting expansion are removed by appropriately selecting successive terms in the power series expansion of the amplitude (or envelope or RG) equation

$$\frac{dA_\epsilon}{d\tau} = a(A_\epsilon, \epsilon).$$

The relationship between *asymptotic matching* or *boundary layer theory* (cf., e.g., Il'in (1992) or O'Malley (1991)) and renormalization can be illustrated by considering the singularly perturbed initial value problem

$$\begin{cases} \dot{x} = xy + \epsilon a x^3, \\ \epsilon \dot{y} = -y + \epsilon b x^2 \end{cases} \tag{78}$$

(introduced in Kuwamura (2001)) on $t \geq 0$ when the constants a and b satisfy $a+b < 0$.

The special case $b = 0$ is of special interest because it is exactly solvable. Because

$$y_\epsilon(t) = e^{-t/\epsilon} y(0),$$

x must be the unique solution

$$x_\epsilon(t) = \frac{e^{\epsilon y(0) e^{-t/\epsilon}} x(0)}{\sqrt{1 - 2a\epsilon x^2(0) \int_0^t e^{\epsilon y(0) e^{-r/\epsilon}} dr}}$$

of the resulting Bernoulli equation. Note that the solution decays algebraically to zero when $a < 0$. It is nearly constant for $a = 0$, and it blows up when

$$\epsilon \int_0^t e^{2\epsilon y(0) e^{-r/\epsilon}} dr = \frac{1}{2ax^2(0)}$$

if $a > 0$.

If we introduce the *fast time*

$$\lambda = \frac{t}{\epsilon}$$

into the corresponding *inner problem*

$$\begin{cases} \dfrac{dx}{d\lambda} = \epsilon xy + \epsilon^2 a x^3, \\ \\ \dfrac{dy}{d\lambda} = -y + \epsilon b x^2, \end{cases}$$

we naturally seek the *inner expansion*

$$\begin{cases} u(\lambda,\epsilon) = u_0(\lambda) + \epsilon u_1(\lambda) + \epsilon^2 u_2(\lambda) + \cdots, \\ v(\lambda,\epsilon) = v_0(\lambda) + \epsilon v_1(\lambda) + \epsilon^2 v_2(\lambda) + \cdots. \end{cases}$$

Proceeding termwise, in the naive manner, we get

$$\begin{aligned} &u_0(\lambda) = x(0), \quad v_0(\lambda) = e^{-\lambda} y(0), \\ &u_1(\lambda) = -(1-e^{-\lambda})x^2(0), \qquad v_1(\lambda) = b(1-e^{-\lambda})x^2(0), \end{aligned}$$

and then, from

$$\frac{du_2}{d\lambda} = u_0 v_1 + u_1 v_0 + a u_0^3$$

and

$$\frac{dv_2}{d\lambda} = -v_2 + 2b u_0 u_1,$$

we get

$$u_2(\lambda) = (a+b)x^3(0)\lambda + bx^3(0)(e^{-\lambda}-1) + \frac{1}{2}(e^{-\lambda}-1)^2 x(0) y^2(0)$$

and

$$v_2(\lambda) = 2bx^2(0)y(0)\left[1 - e^{-\lambda} - \lambda e^{-\lambda}\right].$$

The Tikhonov–Levinson theory applies for t finite and guarantees that the inner expansion can be written as the asymptotic sum

$$\begin{aligned} u(\lambda,\epsilon) &= X(t,\epsilon) + \epsilon\xi(\lambda,\epsilon), \\ v(\lambda,\epsilon) &= \epsilon Y(t,\epsilon) + \eta(\lambda,\epsilon), \end{aligned}$$

where $\binom{X}{\epsilon Y}$ is the *outer expansion* and $\binom{\epsilon\xi}{\eta}$ is the *inner layer correction* that decays to zero exponentially as $\lambda \to \infty$. Replacing λ by t/ϵ defines the surviving outer expansion

$$\begin{aligned} X(t,\epsilon) &= x(0) + \epsilon(x(0)y(0) + (a+b)x^3(0)t) + \epsilon^2(\cdots), \\ \epsilon Y(t,\epsilon) &= \epsilon b x^2(0) + \epsilon^2(\cdots). \end{aligned}$$

Note the secular behavior visible as $\tau = \epsilon t \to \infty$. It is not fixable using Hoppensteadt (1966), because there is no asymptotic stability in t. We can eliminate the secular term, however, by renormalizing. Setting

$$x(0) = A_\epsilon(\tau) + \epsilon P(A_\epsilon(\tau), t, \epsilon) \quad \text{and} \quad y(0) = B_\epsilon(\tau) + \epsilon Q(A_\epsilon(\tau), t, \epsilon)$$

and by picking

$$P_0(A_0, t) = -(a+b)A_0^3 t \quad \text{and} \quad Q_0(A_0, t) = 0,$$

we get a secular-free approximation. Constancy of $x(0)$ and $y(0)$, however, forces A_0 and B_0 to satisfy the limiting amplitude equations

$$\frac{dA_0}{d\tau} = (a+b)A_0^3 \quad \text{and} \quad \frac{dB_0}{d\tau} = 0.$$

This has the algebraically decaying solution

$$A_0(\tau) = \frac{x(0)}{\sqrt{1 - 2(a+b)x^2(0)\tau}}$$

when $a+b < 0$, already observed in the special case $b = 0$. Higher-order terms follow, without difficulty. One could, analogously, also directly seek the asymptotic solution as a function of the three times λ, t, and τ.

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